

# ON DERIVED CATEGORIES AND NONCOMMUTATIVE MOTIVES OF VARIETIES

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## Abstract

In this short note we show how results of Orlov and Toën imply that any equivalence between the derived categories of coherent sheaves on two varieties lifts to an equivalence at the level of dg-categories. This establishes the link between the noncommutative geometry practised by the school of Bondal-Orlov, and the variant developed by Kontsevich and Tabuada. As an application we recover Orlov's result that the derived category determines the Chow motive with rational coefficients up to Tate twists.

## § 1. INTRODUCTION

### 1.1. Overview.

An important invariant of a scheme  $X$  is its bounded derived category  $\mathbf{D}(X)$  of coherent sheaves. In one approach to noncommutative algebraic geometry, the class of spaces under consideration is enlarged by viewing any triangulated category as the derived category of some hypothetical space. This direction has been studied for example by Bondal, Kuznetsov, Lunts, Orlov, Van den Bergh (e.g. [12, 4, 5, 2, 9]).

At the same time triangulated categories have been recognized as poorly behaved for decades: for one example let us note the lack of any kind of homotopy theory of triangulated categories. On the other hand, in nature they arise almost always as *homotopy categories* or truncations of  $(\infty, 1)$ -categories. M. Kontsevich has therefore suggested replacing triangulated categories by stable linear  $(\infty, 1)$ -categories in noncommutative algebraic geometry. Usually pretriangulated dg-categories are taken as a model for stable linear  $(\infty, 1)$ -categories; we will recall some basic notions in section §2.

Lunts-Orlov [10] showed that for a quasi-projective scheme  $X$  over a commutative ring  $k$ , there exists a *unique* dg-category  $\underline{\mathbf{D}}(X)$  whose homotopy category  $H^0(\underline{\mathbf{D}}(X))$  is equivalent to  $\mathbf{D}(X)$  (we say that  $\underline{\mathbf{D}}(X)$  is the unique dg-enhancement of  $\mathbf{D}(X)$ ). Hence given  $X$  it is possible to consider either the triangulated category  $\mathbf{D}(X)$  or the dg-category  $\underline{\mathbf{D}}(X)$ . However it is not clear a priori whether these give the same notion of noncommutative space. Though every dg-functor  $\underline{\mathbf{D}}(X) \rightarrow \underline{\mathbf{D}}(Y)$  induces a triangulated functor  $\mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ , the converse is not obvious: it is in general a very difficult problem to lift triangulated functors. In particular there could be schemes  $X$  and  $Y$  which are isomorphic at the triangulated level, but not at the dg level.

The purpose of this short note is to show that, for smooth proper schemes  $X$  over a field

$k$ , any fully faithful functor  $\mathbf{D}(X) \hookrightarrow \mathbf{D}(Y)$  lifts to a dg-functor. This means that, in passing to the dg-version of noncommutative geometry, at least the notion of isomorphism doesn't change. This fact is an immediate consequence of results of Orlov [12] and Toën [18], but perhaps may not be obvious to non-experts.

Using the theory of noncommutative motives as developed by Tabuada [16] we recover as an immediate corollary the facts that the derived category  $\mathbf{D}(X)$  determines the higher algebraic K-theory, cyclic homology, and Hochschild homology of  $X$ . We also recover a theorem of Orlov [13] stating that the derived category determines the Chow motive up to Tate twists.

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**1.3. Notation.** Throughout the note, we will fix a field  $k$  and work in the category  $\mathbf{SmProj}$  of smooth projective schemes over  $k$ . For  $X, Y \in \mathbf{SmProj}$ , we will write  $X \times Y$  instead of  $X \times_k Y$  for the fibred product over  $k$ .

## § 2. FUNCTORIAL ENHANCEMENT

**2.1.** We will use pretriangulated dg-categories as a model for stable linear  $(\infty, 1)$ -categories. Recall that a *dg-category* (*differential graded category*) over  $k$  is by definition a category enriched over the symmetric monoidal category  $\mathbf{C}(k)$  of complexes of  $k$ -modules. We will write  $\mathbf{DGCat}$  for the category of small dg-categories over  $k$ . Given a dg-category  $\mathcal{A}^\bullet$ , we will write  $H^0(\mathcal{A}^\bullet)$  for its *homotopy category*, given by taking the zeroth cohomologies of all the mapping complexes. By a *dg-enhancement* of a triangulated category  $\mathcal{A}$  we mean a dg-category  $\mathcal{A}^\bullet$  whose homotopy category  $H^0(\mathcal{A}^\bullet)$  is equivalent to  $\mathcal{A}$ . We refer the reader to the introductions of Toën [19] or Keller [7].

Recall that there is a natural notion of equivalence of dg-categories, called *quasi-equivalence*. There is a model structure on  $\mathbf{DGCat}$  where the weak equivalences are the quasi-equivalences (Tabuada [15]), and we let  $\mathbf{DGCat}[\mathcal{W}_{\text{dg}}^{-1}]$  denote the associated homotopy category, that is to say, the localization of  $\mathbf{DGCat}$  at the class of quasi-equivalences.

**2.2.** For two schemes  $X, Y \in \mathbf{SmProj}$ , a *derived correspondence* between  $X$  and  $Y$  is an object of the derived category  $\mathbf{D}(X \times Y)$ . Given derived correspondences  $\mathcal{E}^\bullet \in \mathbf{D}(X \times Y)$  and  $\mathcal{E}'^\bullet \in \mathbf{D}(Y \times Z)$ , one defines their composite as the complex

$$\mathcal{E}'^\bullet \circ \mathcal{E}^\bullet = \mathbf{R}(p_{XZ})_*(\mathbf{L}(p_{XY})^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathbf{L}(p_{YZ})^*(\mathcal{E}'^\bullet))$$

in  $\mathbf{D}(X \times Z)$ , where  $p_{XY}$ ,  $p_{XZ}$  and  $p_{YZ}$  are the projections from  $X \times Y \times Z$ . This defines a category  $\mathbf{SmProj}^{\text{cor}}$  where morphisms are isomorphism classes of derived correspondences.

**2.3.** Let  $\mathbf{TriCat}$  denote the category of small triangulated categories and isomorphism classes of triangulated functors. There is a canonical functor

$$\mathbf{D} : \mathbf{SmProj}^{\text{cor}} \rightarrow \mathbf{TriCat} \quad (2.3.1)$$

which maps a scheme  $X \in \mathbf{SmProj}$  to  $\mathbf{D}(X)$  and a derived correspondence  $\mathcal{E}^\bullet \in \mathbf{D}(X \times Y)$  to the triangulated functor  $\mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  defined by

$$\mathbf{D}(\mathcal{E}^\bullet) := \mathbf{R}(p_Y)_*(\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathbf{L}(p_X)^*(-)),$$

where  $p_X$  and  $p_Y$  are the respective projections from  $X \times Y$ . The functoriality of this construction was shown by Mukai [11].  $\mathbf{D}(\mathcal{E}^\bullet)$  is called the functor *represented* by  $\mathcal{E}^\bullet$ , or the *Fourier-Mukai functor* associated to  $\mathcal{E}^\bullet$ .

**2.4.** By Toën's representability theorem ([18], Theorem 8.15), there are canonical bifunctorial isomorphisms of sets

$$\text{Iso}(\mathbf{D}(X \times Y)) \xrightarrow{\sim} \text{Hom}_{\mathbf{DGCat}[\mathcal{W}_{\text{dg}}^{-1}]}(\underline{\mathbf{D}}(X), \underline{\mathbf{D}}(Y)).$$

Writing  $\mathbf{SmProj}[\mathcal{W}_{\text{dg}}^{-1}]$  for the full subcategory of  $\mathbf{DGCat}[\mathcal{W}_{\text{dg}}^{-1}]$  spanned by the dg-categories  $\underline{\mathbf{D}}(X)$  for  $X \in \mathbf{SmProj}$ , one gets a canonical equivalence of categories

$$\mathbf{SmProj}^{\text{cor}} \xrightarrow{\sim} \mathbf{SmProj}[\mathcal{W}_{\text{dg}}^{-1}] \quad (2.4.1)$$

which is given on objects by  $X \rightsquigarrow \underline{\mathbf{D}}(X)$ .

**2.5.** By Orlov's representability theorem ([12], Theorem 2.2), every fully faithful triangulated functor  $\mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  is represented by some derived correspondence  $\mathcal{E}^\bullet \in \mathbf{D}(X \times Y)$  which is unique up to isomorphism.

By abuse of notation let  $\mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}] \subset \mathbf{TriCat}$  denote the full subcategory spanned by triangulated categories of the form  $\mathbf{D}(X)$  for  $X \in \mathbf{SmProj}$ . Let  $\mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}]_0 \subset \mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}]$  denote the (nonfull) subcategory where the morphisms are only the fully faithful functors. We have a canonical functor

$$\mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}]_0 \longrightarrow \mathbf{SmProj}^{\text{cor}} \quad (2.5.1)$$

which is *faithful* (but not full).

**2.6.** Define the *enhancement functor*  $\varepsilon : \mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}]_0 \rightarrow \mathbf{SmProj}[\mathcal{W}_{\text{dg}}^{-1}]$  as the composite of the functor  $\mathbf{SmProj}[\mathcal{W}_{\text{tri}}^{-1}]_0 \rightarrow \mathbf{SmProj}^{\text{cor}}$  (2.5.1) with the equivalence  $\mathbf{SmProj}^{\text{cor}} \xrightarrow{\sim}$

$\mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}]$  (2.4.1).

$$\begin{array}{ccc} & \mathbf{SmProj}^{\mathrm{cor}} & \\ \nearrow & & \searrow \sim \\ \mathbf{SmProj}[\mathcal{W}_{\mathrm{tri}}^{-1}]_0 & \xrightarrow{\varepsilon} & \mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}] \end{array}$$

This associates to the triangulated category  $\mathbf{D}(X)$  its dg-enhancement  $\underline{\mathbf{D}}(X)$ . Though it is not fully faithful, note that it is conservative (i.e. reflects isomorphisms).

**2.7.** Define the *dehancement functor*  $\delta : \mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}] \rightarrow \mathbf{SmProj}[\mathcal{W}_{\mathrm{tri}}^{-1}]$  as the composite of the equivalence  $\mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}] \xleftarrow{\sim} \mathbf{SmProj}^{\mathrm{cor}}$  with the functor  $\mathbf{SmProj}^{\mathrm{cor}} \rightarrow \mathbf{SmProj}[\mathcal{W}_{\mathrm{tri}}^{-1}]$  induced by  $\mathbf{D} : \mathbf{SmProj}^{\mathrm{cor}} \rightarrow \mathbf{TriCat}$  (2.3.1).

$$\begin{array}{ccc} & \mathbf{SmProj}^{\mathrm{cor}} & \\ \nwarrow \sim & & \searrow \mathbf{D} \\ \mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}] & \xrightarrow{\delta} & \mathbf{SmProj}[\mathcal{W}_{\mathrm{tri}}^{-1}] \end{array}$$

This associates to the dg-category  $\underline{\mathbf{D}}(X)$  its homotopy category  $\mathbf{D}(X)$ .

**2.8. Theorem** (Orlov, Toën). — Let  $X, Y \in \mathbf{SmProj}$  be smooth projective schemes over a field  $k$ . The three conditions

- (i)  $X$  and  $Y$  are isomorphic in  $\mathbf{SmProj}^{\mathrm{cor}}$ .
- (ii) The triangulated categories  $\mathbf{D}(X)$  and  $\mathbf{D}(Y)$  are equivalent.
- (iii) The dg-categories  $\underline{\mathbf{D}}(X)$  and  $\underline{\mathbf{D}}(Y)$  are quasi-equivalent.

are equivalent.

This is an immediate consequence of the existence of the above functors.

### § 3. NONCOMMUTATIVE MOTIVES

**3.1.** Let  $\mathbf{NCSp}$  be the category of noncommutative spaces, i.e. the full subcategory of  $\mathbf{DGCat}[\mathcal{W}_{\mathrm{dg}}^{-1}]$  spanned by smooth proper pretriangulated dg-categories. The category  $\mathbf{NCMot}$  of *noncommutative motives* is the karoubian envelope of the category with the same objects as  $\mathbf{NCSp}$  and where morphisms are Grothendieck groups of internal homs; see Tabuada [16]. The canonical functor

$$\mathbf{U} : \mathbf{NCSp} \longrightarrow \mathbf{NCMot} \quad (3.1.1)$$

is the *universal additive invariant*, i.e. the universal functor sending semi-orthogonal decompositions (of the homotopy category) to direct sums (in some additive category). See (*loc. cit.*,

Theorem 4.2).

**3.2.** For  $X \in \mathbf{SmProj}$ , the dg-category  $\underline{\mathbf{D}}(X)$  is smooth and proper, and hence is a noncommutative space. In particular  $\mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}]$  is a full subcategory of  $\mathbf{NCSp}$ . The noncommutative motive of  $X$ , which we will denote  $\mathrm{NM}(X)$ , is defined as the noncommutative motive of its associated noncommutative space  $\underline{\mathbf{D}}(X)$ .

**3.3. Corollary.** — *Let  $X, Y \in \mathbf{SmProj}$  be smooth projective schemes over a field  $k$ . If the triangulated categories  $\mathbf{D}(X)$  and  $\mathbf{D}(Y)$  are equivalent, then the noncommutative motives  $\mathrm{NM}(X)$  and  $\mathrm{NM}(Y)$  are isomorphic.*

*Proof.* — The enhancement functor  $\varepsilon : \mathbf{SmProj}[\mathcal{W}_{\mathrm{tri}}^{-1}]_0 \rightarrow \mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}]$  lifts a triangulated equivalence  $\mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$  to an isomorphism  $\underline{\mathbf{D}}(X) \xrightarrow{\sim} \underline{\mathbf{D}}(Y)$  in  $\mathbf{SmProj}[\mathcal{W}_{\mathrm{dg}}^{-1}]$ , and therefore in  $\mathbf{NCSp}$ . Hence  $U : \mathbf{NCSp} \rightarrow \mathbf{NCMot}$  gives an isomorphism  $\mathrm{NM}(X) \xrightarrow{\sim} \mathrm{NM}(Y)$ .

**3.4.** Let  $\mathrm{Ho}(\mathbf{Spt})$  denote the homotopy category of spectra. By work of Blumberg-Mandell [3], Keller [6], Schlichting [14] and Thomason-Trobaugh [17] (cf. Tabuada [16]), each of the following can be defined as functors  $\mathbf{NCSp} \rightarrow \mathrm{Ho}(\mathbf{Spt})$  and are additive invariants in the above sense:

- (i) algebraic K-theory,
- (ii) cyclic homology,
- (iii) topological cyclic homology,
- (iv) Hochschild homology,
- (v) topological Hochschild homology.

By universality of the functor  $U : \mathbf{NCSp} \rightarrow \mathbf{NCMot}$  (3.1.1), one has the following corollary.

**Corollary.** — *Let  $X, Y \in \mathbf{SmProj}$  be smooth projective schemes over a field  $k$  and let  $H_*$  denote one of the above functors. If the triangulated categories  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$  are equivalent, then  $H_*(X)$  and  $H_*(Y)$  are isomorphic.*

## § 4. CHOW MOTIVES

**4.1.** Let  $\mathcal{A}$  be an additive category and let  $T : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$  be an auto-equivalence. The *orbit category of  $\mathcal{A}$  with respect to  $T$*  is the category  $\mathcal{A}/T$  whose objects are the same as those of  $\mathcal{A}$  and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}/T}(X, Y) = \bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(X, T^i(Y))$$

for all  $X, Y \in \mathcal{A}$ . The law of composition is defined as follows: for two morphisms  $f = (f^i)_i : X \rightarrow Y$  and  $g = (g^j)_j : Y \rightarrow Z$  in  $\mathcal{A}/T$ , the composite  $g \circ f$  is defined as the morphism whose

$k$ -th component is the sum

$$(g \circ f)^k = \sum_{i+j=k} T^i(g^j) \circ f^i.$$

Let  $\pi_{\mathcal{A}/T} : \mathcal{A} \rightarrow \mathcal{A}/T$  denote the canonical functor which maps a morphism  $f$  to the morphism which is  $f$  in the zeroth component and 0 everywhere else.

**Lemma.** — *Let  $\mathcal{A}$  be an additive category and  $T : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$  an additive auto-equivalence. Suppose that  $\mathcal{A}$  admits arbitrary direct sums. Then the projection functor  $\pi = \pi_{\mathcal{A}/T} : \mathcal{A} \rightarrow \mathcal{A}/T$  admits a right adjoint*

$$\tau = \tau_{\mathcal{A}/T} : \mathcal{A}/T \longrightarrow \mathcal{A}$$

which maps an object  $X$  to the direct sum of all the objects  $T^i(X)$  ( $i \in \mathbf{Z}$ ).

**4.2.** Let  $\mathbf{ChMot}(\mathbf{Q})$  be the category of Chow motives with rational coefficients, and  $M : \mathbf{SmProj}^{\text{op}} \rightarrow \mathbf{ChMot}(\mathbf{Q})$  the canonical functor (see André [1]). Let  $\mathbf{Q}(1) \in \mathbf{ChMot}(\mathbf{Q})$  denote the Tate motive, and recall that the functor  $\mathbf{Q}(1) \otimes - : \mathbf{ChMot}(\mathbf{Q}) \rightarrow \mathbf{ChMot}(\mathbf{Q})$  is an auto-equivalence. Let  $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1)$  denote the associated orbit category; we call this the *category of Chow motives modulo Tate twists*. We will write  $M(i) = M \otimes \mathbf{Q}(i) = M \otimes \mathbf{Q}(1)^{\otimes i}$  for a motive  $M \in \mathbf{ChMot}(\mathbf{Q})$ .

Let  $\mathbf{DM}_{\text{gm}}(\mathbf{Q})$  be the triangulated category of geometric motives of Voevodsky and recall that there is a canonical functor

$$\mathbf{ChMot}(\mathbf{Q}) \longrightarrow \mathbf{DM}_{\text{gm}}(\mathbf{Q}) \tag{4.2.1}$$

which sends the Tate motive  $\mathbf{Q}(1)$  to  $\mathbf{Q}(-1)[-2]$  (where  $-[n]$  denotes the  $n$ -fold composition of the translation functor); see (*loc. cit.*, Théorème 18.3.1.1). Hence one gets an induced functor  $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \longrightarrow \mathbf{DM}_{\text{gm}}(\mathbf{Q})/\mathbf{Q}(-1)[-2]$  on the orbit categories and by the above lemma (4.1), as the triangulated category  $\mathbf{DM}_{\text{gm}}(\mathbf{Q})$  admits arbitrary direct sums, one gets a canonical functor

$$\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \longrightarrow \mathbf{DM}_{\text{gm}}(\mathbf{Q})/\mathbf{Q}(-1)[-2] \xrightarrow{\tau} \mathbf{DM}_{\text{gm}}(\mathbf{Q}) \tag{4.2.2}$$

which maps a motive to the direct sum of all its Tate twists:

$$M \rightsquigarrow \bigoplus_{i \in \mathbf{Z}} M(i)[2i].$$

**4.3.** Kontsevich [8] noted that there is a canonical fully faithful functor

$$\nu : \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \hookrightarrow \mathbf{NCMot}(\mathbf{Q})$$

which is given on morphisms by

$$\bigoplus_i \mathrm{CH}^i(X \times Y, \mathbf{Q}) \xrightarrow{\sim} K_0(X \times Y) \otimes \mathbf{Q},$$

the inverses of the Grothendieck-Riemann-Roch isomorphisms  $\mathrm{ch}(-) \cdot \sqrt{\mathrm{td}_{X \times Y}}$ . Note that this fits into a commutative diagram

$$\begin{array}{ccc} \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}] & \hookrightarrow & \mathbf{NCSp} \\ \downarrow \mathrm{M} & & \downarrow \mathrm{U} \\ \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) & \xhookrightarrow{\nu} & \mathbf{NCMot}(\mathbf{Q}). \end{array}$$

**4.4. Theorem (Orlov [13]).** — *Let  $X, Y \in \mathbf{SmProj}$  be smooth projective schemes over a field  $k$ . If the triangulated categories  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$  are equivalent, then there is an isomorphism*

$$\bigoplus_{i \in \mathbf{Z}} \mathrm{M}(X)(i)[2i] \xrightarrow{\sim} \bigoplus_{j \in \mathbf{Z}} \mathrm{M}(Y)(j)[2j]$$

*in the triangulated category  $\mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$  of geometric motives.*

*Proof.* — Since  $\mathrm{NM}(X) \simeq \mathrm{NM}(Y)$  in  $\mathbf{NCMot}(\mathbf{Q})$  by (3.3), and the functor  $\nu : \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \hookrightarrow \mathbf{NCMot}(\mathbf{Q})$  is fully faithful, one has  $\mathrm{M}(X) \simeq \mathrm{M}(Y)$  in  $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1)$ . Then the functor  $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \rightarrow \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$  (4.2.2) gives the desired isomorphism.

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